Clairaut's Theorem

10/4/21

If $f(x,y)$ has cts. second-order partial derivatives on an open disk D, then $\frac{d^2f}{dxdy} = \frac{d^2f}{dydx}$ on D.

Notation: $f_x = \frac{df}{dx}$, $f_y = \frac{df}{dy}$

$f_{xx} = \frac{d^2f}{(dx)^2}$

$f_{xy} = (f_x)_y = \frac{d}{dy}\left(\frac{d}{dx}(f)\right) = \frac{d^2f}{dydx}$

Pf: Let $f(x,y)$ has cts. second-order mixed partial derivatives on some open disk D and suppose $(a,b) \in D$.

Let $\Delta(h) := \left(f(a+h, b+h) - f(a+h, b)\right) - \left(f(a, b+h) - f(a,b)\right)$

for all $h \neq 0$ where $(a+h, b+h)\ (a+h, b)\ (a, b+h) \in D$

Let $\alpha(x) := f(x, b+h) - f(x, b)$ and notice $\Delta(h) = \alpha(a+h) - \alpha(a)$

For h fixed, we can apply the MVT to obtain c_h satisfying $|a - c_h| \leq |h|$ and $\alpha'(c_h)h = \alpha(a+h) - \alpha(a)$. Thus
$$\Delta(h) = \alpha(a+h) - \alpha(a) = h\alpha'(c_h) = h\left(f_x(c_h, b+h) - f_x(c_h, b)\right)$$

Let $B(y) := f_x(c_h, y)$, we see again by MVT there is d_h satisfying $|b - d_h| \leq |h|$ and $B'(d_h)h = f_x(c_h, b+h) - f_x(c_h, b)$

Thus $\Delta(h) = h\left(f_x(c_h, b+h) - f_x(c_h, b)\right) = h(hB'(d_h)) = \underbrace{h^2 f_{xy}(c_h, d_h)}_{*}$.

If we rearrange $\Delta(h) = \big(f(a+h, b+h) - f(a, b+h)\big) - \big(f(a+h, b) - f(a,b)\big)$, we can repeat the argument (using y first) to obtain γ_h, δ_h satisfying $|a - \gamma_h| \le |h|$, $|b - \delta_h| \le |h|$ and $\Delta(h) = h^2 f_{yx}(\gamma_h, \delta_h)$ for all fixed h.

(**)

Notice $\lim_{h \to 0} (c_h, d_h) = (a,b) = \lim_{h \to 0} (\gamma_h, \delta_h)$ (by construction)

Thus we compute:

$$f_{xy}(a,b) = f_{xy}\left(\lim_{h\to 0}(c_h, d_h)\right)$$

By continuity $= \lim_{h \to 0} f_{xy}(c_h, dh)$

by (*) $= \lim_{h\to 0} \frac{\Delta(h)}{h^2}$

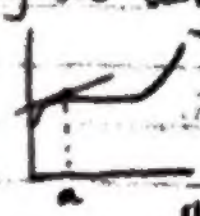
by (**) $= \lim_{h \to 0} f_{yx}(\gamma_h, \delta_h)$

$= f_{yx} \lim_{h\to 0} (\gamma_h, \delta_h)$

$= f_{yx}(a,b)$. ▨

§14.? Linear Approximation of Multivariable Functions

IDEA: In Calc I, we say the tangent line to f at a "well-approximates" f near $(a, f(a))$.



as $x \to a$ the error approximating f w/ the tangent line goes to 0.

In Calc III we use a tangent (hyper) plane instead (since taylor approximating)

Small changes in input have change for output of f measured by the first derivatives. Calc I - $f(x) \approx y = f(a) + f'(a)(x-a)$ near input a

In Calc III, these changes are measured by:

$$f(x,y) \approx z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{near } (a,b)$$

Ex. Find an equation of the tangent plane to $f(x,y) = x^2 + xy - y^2$ at $(4,1)$

Sol. The tangent plane has equation $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(4,1) = 4^2 + 4 \cdot 1 - 1^2 = 19$

$f_x(x,y) = 2x + y$ $\qquad$ $f_y(x,y) = x - 2y$

$f_x(4,1) = 2 \cdot 4 + 1 = 9$ $\qquad$ $f_y(4,1) = 4 - 2 = 2$

Therefore the desired plane is $z = 19 + 9(x-4) + 2(y-1)$ ▨

Ex. Find tangent plane to $f(x,y) = \frac{e^{x-y}}{x}$ at $(2, 2, \frac{1}{2})$.

Sol. $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(2,2) = \frac{1}{2}$

$f_x(x,y) = \frac{xe^{x-y} - e^{x-y}}{x^2}$, $\qquad$ $f_y(x,y) = \frac{-e^{x-y}}{x}$

$f_x(2,2) = \frac{1}{4}$ $\qquad$ $f_y(2,2) = -\frac{1}{2}$

$z = \frac{1}{2} + \frac{1}{4}(x-2) - \frac{1}{2}(y-2)$ ▨

In Calc I, we also thought from the perspective of differentials

$\Delta f \approx f'(a) \Delta x$ at $a = x$

changing f from change in x $\qquad$ change in x

For functions with 2 variables

$$\Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

In calc I, Δ replaced by symbols and inserted equalities

$$df = f'(x)\,dx \quad \text{ie} \quad df = \frac{df}{dx}dx$$

__Defn:__ The __total differential__ of function f, of variable x_1 to x_n is

$$df = \frac{df}{dx_1}dx_1 + \frac{df}{dx_2}dx_2 + \dots + \frac{df}{dx_n}dx_n$$

represents change in x

represent loose changes in x

Ex. Compute total differential of $F(x,y,z) = \frac{\ln(x-3y)}{z}$

Sol. Compute

$$f_x(x,y,z) = \frac{1}{z}\cdot\frac{1}{x-3y} = \frac{1}{(x-3y)z} \qquad f_y(x,y,z) = \frac{-3}{(x-3y)z}$$

$$f_z(x,y,z) = -\frac{\ln(x-3y)}{z^2}$$

$$df = f_x\,dx + f_y\,dy + f_z\,dz = \frac{1}{z(x-3y)}dx - \frac{3}{z(x-3y)}dy - \frac{\ln(x-3y)}{z^2}dz \quad \blacksquare$$

Ex. Estimate change in F from $(4,1,1)$ to $(4.5, 1.5, .5)$

Sol. $$\Delta f \approx df = \frac{1}{z(x-3y)}dx - \frac{3}{z(x-3y)}dy - \frac{\ln(x-3y)}{z^2}dz$$

$$\Delta f \approx f_x(4,1,1)\Delta x + f_y(4,1,1)\Delta y + f_z(4,1,1)\,dz$$

$$= 1\,(4.5-4) - 3\,(1.5-1) - \frac{\ln(1)}{1}(.5-1)$$

$$= \frac{1}{2} - \frac{3}{2} - 0 = -1$$

$\blacksquare$